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COMMENT

The gap exponent Δ for the spin-*s* Ising model on the FCC lattice

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Abstract. High-temperature series expansions are presented for the magnetisation $M(K, h)$ of the spin-s Ising model on the face-centred cubic lattice. Series coefficients are derived to order 13 and 5 respectively in the variables $K = J/kT$ and $h = mH/kT$, for $\frac{1}{2} \leq s \leq \frac{9}{2}$. The data are analysed for $2\Delta - \gamma$, where Δ and γ denote the gap exponent and the susceptibility exponent respectively. The estimate $2\Delta - \gamma = 1.890 \pm 0.003$ satisfies the data for all s.

The validity of hyperscaling for the spin-s Ising model has been the subject of much recent investigation (Baker 1977, Nickel 1982a, b, Roskies 1981a, b, Gaunt 1982 and references therein). Of particular interest is the hyperscaling relation,

$$d\nu = 2\Delta - \gamma, \tag{1}$$

where d is the spatial dimensionality and γ, ν and Δ denote the critical exponents characterising the high-temperature susceptibility, correlation length and successive field derivatives of the free energy. High-temperature series expansions have been used to estimate these critical exponents and recent work on the spin-$\frac{1}{2}$ model (Baker 1977, Nickel 1982a, b, Roskies 1981a, b, Zinn-Justin 1981) shows some disagreement among workers about the validity of (1). Previous work on the general spin model has been confined to the exponents γ and ν (Camp and Van Dyke 1975, Saul *et al* 1975, Camp *et al* 1976 and Nickel 1982a).

In this paper, we investigate the exponent Δ for the spin-s Ising model on the face-centred cubic (FCC) lattice. The necessary series expansions are derived for $\frac{1}{2} \leq s \leq \frac{9}{2}$. Estimates for $2\Delta - \gamma$ (the right-hand side of equation (1)) are obtained by Padé approximant techniques (Gaunt and Guttmann 1974), using critical point renormalisation (Baker 1977).

The linked cluster expansion (Wortis 1974) in vertex renormalised form was used to calculate the magnetisation $M(K, h)$ to order 13 and 5 in the expansion variables $K = J/kT$ and $h = mH/kT$. Derivatives of M with respect to h yield all the series expansions needed to estimate $2\Delta - \gamma$. Thus,

$$M = \partial \ln Z/\partial h = \sum_n K^n \sum_m b_{mn} h^m, \tag{2}$$

$$\chi_0 = \partial M/\partial h|_{h=0} \sim (1 - K/K_c)^{-\gamma}, \tag{3}$$

$$\chi_0^{(2)} = \partial^4 \ln Z/\partial h^4|_{h=0} = \partial^3 M/\partial h^3|_{h=0} \sim (1 - K/K_c)^{-\gamma-2\Delta}, \tag{4}$$

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Table 1. $s = \frac{1}{2}$ Ising model. Coefficients of the magnetisation series. Expansion variables are $h = mH/kT$ and $K = J/kT$.

n \ m	1	3	5
1	11.999 999 999	−15.999 999 999	13.599 999 999
2	131.999 999 999	−451.999 999 999	701.599 999 999
3	1 399.999 999 999	−9 866.666 666 666	25 338.666 666 666
4	14 563.999 999 999	−184 366.666 666 666	730 125.866 666 666
5	149 713.599 999 999	−3 103 106.133 333 333	17 973 531.413 333 333
6	1 526 845.866 666	−48 443 542.755 555	393 775 905.848 888
7	15 483 628.952 380	−714 520 977.269 841	7 884 806 754.946 031
8	156 350 472.419 047	−10 081 215 372.634 920	146 966 260 843.027 301
9	1 573 541 965.214 814	−137 251 331 375.418 694	2 583 799 624 938.777 848
10	15 794 123 458.431 322	−1 814 655 363 184.207 689	43 269 596 399 483.163 541
11	158 183 042 921.275 228	−23 411 605 670 190.974 378	695 436 555 443 004.811 184
12	1 581 353 999 968.296 860	−295 827 433 384 150.023 237	10 790 510 478 217 485.389 384
13	15 784 274 098 318.319 199	−3 671 859 423 112 001.580 257	162 397 322 558 459 795.259 627

where Z is the partition function and K_c denotes the critical point. The coefficients b_{mn} (equation (2)) are given in tables 1–9 for $\frac{1}{2} \leqslant s \leqslant \frac{9}{2}$.

The exponents γ and Δ can be estimated separately by using Padé or ratio methods on the series (3) and (4). However, it proves more convenient to evaluate $2\Delta-\gamma$ directly by the use of 'critical point renormalisation' (Baker 1977). Denoting the $\chi_0^{(2)}$ series by $f(x)$ and $[\chi_0]^2$ by $g(x)$, we form the function $h(x)$ by

$$f(x)=\sum_i f_i x^i \sim (1-x/x_c)^{-\gamma-2\Delta}, \tag{5}$$

$$g(x)=\sum_i g_i x^i \sim (1-x/x_c)^{-2\gamma}, \tag{6}$$

$$h(x)=\sum_i (f_i/g_i) x^i \sim (1-x)^{-(2\Delta-\gamma+1)}. \tag{7}$$

The asymptotic form shown in (5) and (6) is valid for values of x close to x_c, while that in (7) is valid near $x = 1$.

Table 2. $s = 1.0$ Ising model.

n \ m	1	3	5
1	5.333 333 333	−3.555 555 555	1.599 999 999
2	39.999 999 999	−67.851 851 851	54.888 888 888
3	292.444 444 444	−1 009.580 246 913	1 334.235 390 946
4	2 105.703 703 703	−12 932.395 061 728	26 079.246 090 534
5	15 011.599 999 999	−149 772.572 839 506	437 827.865 459 533
6	106 285.090 534	−1 612 884.538 271	6 566 747.817 954
7	748 748.184 322	−16 439 261.494 258	90 270 726.068 999
8	5 254 538.777 269	−160 487 134.169 243	1 157 594 862.164 014
9	36 763 932.502 594	−1 513 301 661.647 458	14 024 994 328.340 518
10	256 595 680.786 579	−13 867 899 091.417 901	162 070 808 169.900 763
11	1 787 319 915.984 674	−124 083 192 361.910 950	1 799 372 030 240.640 544
12	12 428 583 595.406 453	−1 087 910 010 008.435 633	19 303 118 941 328.309 657
13	86 301 184 251.945 360	−9 373 098 417 305.640 879	201 003 295 138 760.413 003

Table 3. $s = 1.5$ Ising model.

n \ m	1	3	5
1	3.703 703 703	−1.865 569 272	0.651 729 919
2	23.292 181 069	−29.771 681 146	18.573 117 241
3	143.196 463 953	−371.330 025 346	376.555 463 855
4	868.144 422 428	−3 993.646 464 361	6 154.694 109 593
5	5 215.090 096 548	−38 875.542 441 804	86 557.266 555 632
6	31 127.629 188	−352 165.838 319	1 088 933.733 974
7	184 916.176 573	−3 021 216.816 070	12 568 182.470 853

Table 4. $s = 2.0$ Ising model.

$n \backslash m$	1	3	5
1	2.999 999 999	−1.299 999 999	0.396 249 999
2	17.024 999 999	−18.698 749 999	10.144 296 874
3	94.561 249 999	−210.418 541 666	185.100 244 791
4	518.236 953 124	−2 043.174 299 479	2 726.089 709 635
5	2 815.187 722 656	−17 965.299 069 661	34 573.958 732 356
6	15 198.292 550	−147 055.421 055	392 481.065 046
7	81 674.889 423	−1 140 267.657 183	4 089 384.644 135
8	437 368.659 977	−8 474 114.845 338	39 791 577.239 881
9	2 335 550.532 868	−60 859 714.597 609	366 135 931.880 654
10	12 443 468.322 785	−424 953 070.518 203	3 215 570 844.120 880
11	66 171 918.918 841	−2 898 086 715.473 624	27 148 222 268.440 930
12	351 330 252.643 250	−19 372 088 227.898 802	221 577 812 309.807 161
13	1 862 807 383.083 883	−127 276 755 616.126 311	1 756 136 980 083.862 033

Table 5. $s = 2.5$ Ising model.

$n \backslash m$	1	3	5
1	2.613 333 333	−1.031 395 555	0.288 655 359
2	13.861 119 999	−13.856 681 149	6.889 437 821
3	71.999 103 431	−145.721 118 669	117.316 330 485
4	369.129 428 233	−1 322.791 258 819	1 613.469 693 261
5	1 876.190 158 181	−10 876.240 552 227	19 117.560 853 178
6	9 478.396 125	−83 265.592 504	202 817.794 115
7	47 668.698 069	−603 939.279 993	1 975 405.977 443
8	238 902.691 537	−4 198 851.636 113	17 971 492.083 612
9	1 194 012.349 823	−28 213 278.872 067	154 631 000.231 890
10	5 954 158.892 715	−184 324 246.344 836	1 270 070 177.121 516
11	29 636 193.932 182	−1 176 238 935.308 203	10 029 313 672.643 745
12	147 279 452.309 904	−7 357 391 698.033 312	76 569 134 502.544 371
13	730 935 585.996 131	−45 235 257 922.955 234	567 695 217 996.283 600

Table 6. $s = 3.0$ Ising model.

$n \backslash m$	1	3	5
1	2.370 370 370	−0.877 914 951	0.231 741 677
2	11.983 539 094	−11.238 008 143	5.263 932 375
3	59.351 928 059	−112.639 068 925	85.359 444 100
4	290.193 064 839	−974.734 203 899	1 118.379 420 004
5	1 406.810 073 235	−7 641.311 880 896	12 627.367 880 567
6	6 779.142 219	−55 782.456 093	127 680.239 198
7	32 521.847 073	−385 838.179 610	1 185 430.186 967
8	155 481.819 493	−2 558 297.185 984	10 281 494.948 374
9	741 301.335 551	−16 394 768.448 768	84 345 433.125 101
10	3 526 483.973 786	−102 160 681.873 952	660 569 412.866 354
11	16 745 024.588 688	−621 815 400.357 138	4 974 103 966.423 670
12	79 387 555.914 801	−3 709 939 929.587 614	36 213 712 591.705 365
13	375 872 345.928 976	−21 757 431 633.935 554	256 052 791 675.351 964

Table 7. $s = 3.5$ Ising model.

$n \backslash m$	1	3	5
1	2.204 081 632	−0.779 675 135	0.197 226 860
2	10.750 520 616	−9.626 732 301	4.317 885 119
3	51.382 060 196	−93.087 684 577	67.512 338 971
4	242.464 014 381	−777.252 096 076	853.102 208 016
5	1 134.518 042 989	−5 879.745 020 770	9 291.393 988 029
6	5 276.992 796	−41 422.322 856	90 636.531 664
7	23 436.348 387	−276 510.811 870	811 912.410 006
8	112 771.572 677	−1 769 483.692 555	6 794 788.353 966
9	519 016.010 058	−10 944 709.550 811	53 789 063.481 708
10	2 383 412.295 044	−65 826 083.673 739	406 522 898.899 995
11	10 924 925.919 266	−386 723 666.885 002	2 954 154 893.600 217
12	49 999 298.115 664	−2 227 097 926.080 874	20 756 692 552.541 242
13	228 524 782.674 196	−12 607 240 853.649 874	141 642 606 010.815 497

Table 8. $s = 4.0$ Ising model.

$n \backslash m$	1	3	5
1	2.083 333 333	−0.711 805 555	0.174 316 406
2	9.882 812 499	−8.546 242 042	3.709 061 516
3	45.946 234 809	−80.370 193 293	56.378 670 269
4	210.915 282 920	−652.696 407 447	692.702 237 558
5	960.099 969 916	−4 802.674 975 328	7 336.556 368 911
6	4 344.598 553	−32 912.126 571	69 601.489 003
7	19 573.426 807	−213 720.755 701	606 397.172 648
8	87 882.862 089	−1 330 475.651 936	4 936 046.267 968
9	393 517.596 167	−8 005 721.517 832	38 007 615.006 719
10	1 758 187.184 526	−46 842 308.053 558	279 414 991.453 365
11	7 840 970.927 619	−267 726 739.418 730	1 975 141 347.976 270
12	34 914 268.722 607	−1 499 983 960.225 922	13 499 991 992.387 603
13	155 260 830.992 910	−8 260 918 694.281 715	89 616 581 265.933 498

Table 9. $s = 4.5$ Ising model.

$n \backslash m$	1	3	5
1	1.991 769 547	−0.662 283 865	0.158 107 016
2	9.240 827 109	−7.776 014 839	3.288 607 369
3	42.022 125 821	−71.518 287 221	48.874 504 042
4	188.694 343 450	−568.070 180 019	587.201 871 517
5	840.244 533 741	−4 088.492 310 766	6 081.960 847 664
6	3 719.522 428	−27 405.669 370	56 429.683 804
7	16 393.034 426	−174 079.359 742	480 844.487 981
8	72 003.956 749	−1 060 065.469 338	3 828 258.377 912
9	315 412.931 782	−6 239 626.939 927	28 832 354.571 114
10	1 378 625.370 194	−35 713 684.405 971	207 327 640.402 512
11	6 014 768.061 180	−199 678 423.789 039	1 433 548 586.191 700
12	26 201 220.686 570	−1 094 392 273.732 577	9 584 330 973.230 610
13	113 985 957.862 072	−5 896 103 869.178 838	62 235 320 948.249 463

Table 10. $2\Delta-\gamma$ for the spin-s Ising model.

s	$2\Delta-\gamma$	s	$2\Delta-\gamma$
$\frac{1}{2}$	1.8873 ± 0.0005	3	1.8926 ± 0.001
1	1.8904 ± 0.001	$\frac{7}{2}$	1.8927 ± 0.001
$\frac{3}{2}$	1.8915 ± 0.001	4	1.8928 ± 0.001
2	1.8921 ± 0.001	$\frac{9}{2}$	1.8928 ± 0.001
$\frac{5}{2}$	1.8925 ± 0.001		

The quantity $2\Delta-\gamma$ can be evaluated by forming Padé approximants to the logarithmic derivatives of $h(x)$. Thus,

$$(1-x)\,\mathrm{d}\ln h(x)/\mathrm{d}x|_{x=1} = -(2\Delta-\gamma+1). \tag{8}$$

The results for $\frac{1}{2}\leq s\leq\frac{9}{2}$ are presented in table 10. The estimate of 1.8873 ± 0.0005 for $s=\frac{1}{2}$ is very close to that obtained by Baker (1977) using shorter series. The corresponding values for $s>\frac{1}{2}$ appear to increase very slowly with s. An estimate of 1.890 ± 0.003 covers the entire range of values, and (1) now yields

$$\nu = 0.630\pm0.001. \tag{9}$$

This is significantly lower than the estimates of Camp *et al* (1976) and Baker (1977) from correlation length series, but very close to renormalisation group predictions (Le Guillou and Zinn-Justin 1977) and the series estimates of Roskies (1981b), Zinn-Justin (1981) and Nickel (1982b).

The analysis presented here suggests that the estimate $2\Delta-\gamma=1.890\pm0.003$ is satisfied for all s. The question of the validity of (1) still depends on the value assigned to ν. Further extrapolation studies based on extended series expansions on the FCC lattice are in progress and will be reported in due course.

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